

A Proof of Fejes Tóth's Conjecture on Sphere Packings with Kissing Number Twelve

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1 Fejes Tóth's Conjecture

On December 26, 1994, L. Fejes Tóth wrote to me, “I suppose that you will be interested in the following conjecture: In 3-space any packing of equal balls such that each ball is touched by twelve others consists of hexagonal layers. In the enclosed papers a strategy is described to prove this conjecture” [Fej89], [Fej69]. This article verifies Fejes-Tóth's conjecture.¹

A packing of balls in this article is identified with its set of centers. We adopt the convention that balls in a packing have unit radius. Formally, a *packing* is a set $V \subset \mathbb{R}^3$ for which $\|\mathbf{v} - \mathbf{w}\| < 2$ implies $\mathbf{v} = \mathbf{w}$, for every $\mathbf{v}, \mathbf{w} \in V$. It is known that the kissing number in three dimensions is twelve [Lee56]. Call a nonempty packing V in \mathbb{R}^3 in which every $\mathbf{v} \in V$ has distance 2 from twelve other $\mathbf{w} \in V$ a *packing with kissing number twelve*.

Two examples of packings consisting of hexagonal layers in which each ball is touched by twelve others are the face-centered cubic packing (FCC) and the hexagonal-close packing (HCP). In the FCC packing, the arrangement of twelve other balls around each ball is identical. We call this particular arrangement of twelve balls the FCC pattern. Similarly, in the HCP packing, the arrangement of twelve around each ball is identical, and we call this particular arrangement the HCP pattern.

It is well known that if the arrangement around each ball is either the FCC or HCP pattern, then the packing consists of hexagonal layers [Hal12a, Sec. 1.3]. Hence the following theorem, which is the main result of this article, is enough to guarantee that a packing with kissing number twelve consists of hexagonal layers.

Theorem 1 (Packings with kissing number twelve). *Let V be a packing with kissing number twelve. Then for every point $\mathbf{u} \in V$, the set of twelve around that point is arranged in the pattern of the HCP or FCC packing.*

The truncation parameter $h_0 = 1.26$ will be used throughout this article. The following estimate is one of the main results of [Hal12a]. Its proof is omitted.

Lemma 1. *Let V be a finite packing. Assume that $2 \leq \|\mathbf{v}\| \leq 2h_0$ for every $\mathbf{v} \in V$. Let $L(h) = (h_0 - h)/(h_0 - 1)$. Then*

$$\sum_{\mathbf{v} \in V} L(\|\mathbf{v}\|/2) \leq 12. \quad (1)$$

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¹ The results of this article were presented at the Fields Institute, September 2011.

Lemma 2. *Let V be any packing with kissing number twelve and let $\mathbf{u}, \mathbf{v} \in V$. Then $\mathbf{u} = \mathbf{v}$, $\|\mathbf{u} - \mathbf{v}\| = 2$, or $\|\mathbf{u} - \mathbf{v}\| \geq 2h_0$.*

Proof. Let $\mathbf{u}_1, \dots, \mathbf{u}_{12}$ be the twelve kissing points around \mathbf{u} ; that is, $\|\mathbf{u}_i - \mathbf{u}\| = 2$. By translating V , we may assume without loss of generality that $\mathbf{u} = \mathbf{0}$. Assume that $\mathbf{v} \neq \mathbf{u}_i$, $\mathbf{0}$ is in V . By Lemma 1,

$$L(\|\mathbf{v}\|/2) + 12 = L(\|\mathbf{v}\|/2) + \sum_{i=1}^{12} L(\|\mathbf{u}_i\|/2) \leq 12.$$

This implies that $L(\|\mathbf{v}\|/2) \leq 0$, so $\|\mathbf{v}\| \geq 2h_0$. \square

A packing V may always be translated so that $\mathbf{0} \in V$. We study the structure of a kissing configuration centered at the origin that has the separation property of Lemma 2.

Definition 1 ($S^2(2), \mathcal{V}$). *Let $S^2(2)$ be the sphere of radius 2, centered at the origin. Let \mathcal{V} be the set of packings $V \subset \mathbb{R}^3$ such that*

1. $\text{card}(V) = 12$,
2. $V \subset S^2(2)$,
3. *For all $\mathbf{u}, \mathbf{v} \in V$, we have $\mathbf{u} = \mathbf{v}$, $\|\mathbf{u} - \mathbf{v}\| = 2$, or $\|\mathbf{u} - \mathbf{v}\| \geq 2h_0$.*

If $V \in \mathcal{V}$, set

$$E_2(V) = \{\{\mathbf{v}, \mathbf{w}\} : \|\mathbf{v} - \mathbf{w}\| = 2\},$$

the set of *contact edges*.

2 Definitions and Review

We follow the general approach to sphere packing problems described in [Hal12a]. We use a number of definitions given there. In particular, we have the following.

Definition 2 (affine). *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ and $S' = \{\mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ are finite subsets of \mathbb{R}^N , then set*

$$\text{aff}_{\pm}(S, S') = \{t_1 \mathbf{v}_1 + \dots + t_n \mathbf{v}_n : t_1 + \dots + t_n = 1, \pm t_j \geq 0, \text{ for } j > k\},$$

$$\text{aff}_{\pm}^0(S, S') = \{t_1 \mathbf{v}_1 + \dots + t_n \mathbf{v}_n : t_1 + \dots + t_n = 1, \pm t_j > 0, \text{ for } j > k\}.$$

This notation is general enough to describe rays, lines, open and closed intervals, convex hulls, and affine hulls. To lighten the notation for singleton sets, abbreviate $\text{aff}_{\pm}(\{\mathbf{v}\}, S')$ to $\text{aff}_{\pm}(\mathbf{v}, S')$. If $S \subset \mathbb{R}^3$ is a finite set of points, abbreviate

$$C_{\pm}(S) = \text{aff}_{\pm}(\mathbf{0}, S),$$

$$C_{\pm}^0(S) = \text{aff}_{\pm}^0(\mathbf{0}, S).$$

When the subscript is absent, the subscript $+$ is implied: $C_+(S) = C(S)$, and so forth. The parentheses around the set are frequently omitted without change in meaning:

$$C^0\{\mathbf{v}, \mathbf{w}\} = C_+^0(\{\mathbf{v}, \mathbf{w}\}) = \text{aff}_{+}^0(\{\mathbf{0}\}, \{\mathbf{v}, \mathbf{w}\}) = \text{aff}_{+}^0(\mathbf{0}, \{\mathbf{v}, \mathbf{w}\}).$$

Definition 3 (fan, blade). Let (V, E) be a pair consisting of a set $V \subset \mathbb{R}^3$ and a set E of unordered pairs of distinct elements of V . The pair is said to be a fan if the following properties hold.

1. (CARDINALITY) V is finite and nonempty.
2. (ORIGIN) $\mathbf{0} \notin V$.
3. (NONPARALLEL) If $\{\mathbf{v}, \mathbf{w}\} \in E$, then the set $\{\mathbf{0}, \mathbf{v}, \mathbf{w}\}$ is not collinear.
4. (INTERSECTION) For all $\varepsilon, \varepsilon' \in E \cup \{\{\mathbf{v}\} : \mathbf{v} \in V\}$,

$$C(\varepsilon) \cap C(\varepsilon') = C(\varepsilon \cap \varepsilon').$$

When $\varepsilon \in E$, call $C^0(\varepsilon)$ or $C(\varepsilon)$ a blade of the fan.

Basic properties of fans are developed in [Hal12a, Ch. 5]. Every fan is graph with vertex set V and edge set E . We use the terminology of graph theory to describe fans. For example, we say that \mathbf{v}, \mathbf{w} are *adjacent* if $\{\mathbf{v}, \mathbf{w}\} \in E$.

Definition 4 (hypermap). A hypermap is a finite set D , together with three functions $e, n, f : D \rightarrow D$ that compose to the identity

$$enf = I_D.$$

The elements of D are called darts. The functions e, n and f are called the edge map, the node map, and the face map, respectively. The orbits of the set of darts under the edge map, node map, and face map are called edges, faces, and nodes of the hypermap.

Example 1 (dihedral). There is a hypermap Dih_{2k} with a dart set of cardinality $2k$. The permutations f, n, e have orders $k, 2$, and 2 respectively, and $enf = I$. The set of darts is given by

$$\{x, fx, f^2x, \dots, f^{k-1}x\} \cup \{nx, nfx, nf^2x, \dots, nf^{k-1}x\}$$

for any dart x . If a hypermap is isomorphic to Dih_{2k} for some k , then it is *dihedral*. (The three permutations generate the dihedral group of order $2k$, acting on a set of $2k$ darts under the left action of the group upon itself.)

Basic properties of hypermaps are developed in [Hal12a, Ch. 4]. An *isomorphism of hypermaps* is a bijection $\phi : D \mapsto D'$ such that $\phi(hx) = h'\phi x$, for all $x \in D$ and all structure permutations $(h, h') = (e, e'), (f, f'), (n, n')$.

For each $\mathbf{v} \in V$ such that $\text{card}(E(\mathbf{v})) > 1$, we define a cyclic permutation $\sigma(\mathbf{v}) : E(\mathbf{v}) \rightarrow E(\mathbf{v})$ as follows. Choose $\pi_{\mathbf{v}} : \mathbb{R}^3 \rightarrow \mathbb{C}$, a surjective real-linear map, with $\mathbf{v} \in \ker \pi_{\mathbf{v}}$, chosen so that $(\pi_{\mathbf{v}}^{-1}(1), \pi_{\mathbf{v}}^{-1}(\sqrt{-1}), \mathbf{v}/\|\mathbf{v}\|)$ is a positively-oriented, orthonormal basis of \mathbb{R}^3 . Define the permutation $\sigma(\mathbf{v})$ by pulling back the counterclockwise cyclic permutation of the points on the unit circle:

$$\{\pi_{\mathbf{v}}(\mathbf{w})/\|\pi_{\mathbf{v}}(\mathbf{w})\| : \mathbf{w} \in E(\mathbf{v})\} \subset \mathbb{C}^\times.$$

Write $\sigma(\mathbf{v}, \mathbf{w})$ for $\sigma(\mathbf{v})(\mathbf{w}) \in E(\mathbf{v})$. Let $\arg(re^{i\theta}) = \theta \in [0, 2\pi)$ be the argument of a complex number, and define the azimuth angle by

$$\text{azim}(\mathbf{0}, \mathbf{v}, \mathbf{u}, \mathbf{w}) = \arg(\pi_{\mathbf{v}}(\mathbf{w})/\pi_{\mathbf{v}}(\mathbf{u})).$$

When $x = (\mathbf{v}, \mathbf{w}) \in D_1$, set

$$\text{azim}(x) = \text{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}, \sigma(\mathbf{v}, \mathbf{w})).$$

Also, set

$$W_{\text{dart}}^0(x) = \{\mathbf{p} \in \mathbb{R}^3 : 0 < \text{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}, \mathbf{p}) < \text{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}, \sigma(\mathbf{v}, \mathbf{w}))\}.$$

We may associate a hypermap with a fan by the following construction. Let (V, E) be a fan. Define a set of darts D to be the disjoint union of two sets D_1, D_2 :

$$\begin{aligned} D_1 &= \{(\mathbf{v}, \mathbf{w}) : \{\mathbf{v}, \mathbf{w}\} \in E\}, \\ D_2 &= \{(\mathbf{v}, \mathbf{v}) : \mathbf{v} \in V, E(\mathbf{v}) = \emptyset\}, \text{ and} \\ D &= D_1 \cup D_2, \end{aligned}$$

where $E(\mathbf{v}) = \{\mathbf{w} : \{\mathbf{v}, \mathbf{w}\} \in E\}$, the set of darts adjacent to \mathbf{v} . Darts in D_2 are said to be *isolated* and darts in D_1 are *nonisolated*. Define permutations n, e , and f on D_1 by

$$\begin{aligned} n(\mathbf{v}, \mathbf{w}) &= (\mathbf{v}, \sigma(\mathbf{v}, \mathbf{w})), \\ f(\mathbf{v}, \mathbf{w}) &= (\mathbf{w}, \sigma(\mathbf{w})^{-1}\mathbf{v}), \\ e(\mathbf{v}, \mathbf{w}) &= (\mathbf{w}, \mathbf{v}). \end{aligned}$$

Define permutations n, e, f on D_2 by making them degenerate on D_2 :

$$n(\mathbf{v}) = e(\mathbf{v}) = f(\mathbf{v}) = \mathbf{v}.$$

Set $\text{hyp}(V, E) = (D, e, n, f)$. If (V, E) is a fan, then it is known that $\text{hyp}(V, E)$ is a hypermap [Hal12a].

If (V, E) is a hypermap, then we define $X(V, E)$ to be the union of the sets $C(\varepsilon)$, for $\varepsilon \in E$, and $Y(V, E) = \mathbb{R}^3 \setminus X(V, E)$ to be its complement. There is a well-defined mapping, $F \rightarrow U_F$, from the set of faces of $\text{hyp}(V, E)$ to the set of topological connected components of $Y(V, E)$. The component U_F is characterized by the condition

$$U_F \cap B(\mathbf{v}, \epsilon) \cap W_{\text{dart}}^0(x) \neq \emptyset,$$

for all $x = (\mathbf{v}, \mathbf{w}) \in F$ and all sufficiently small $\epsilon > 0$ (where $B(\mathbf{v}, \epsilon)$ is an open ball of radius ϵ at \mathbf{v}). We write $\text{sol}(U_F)$ for the solid angle of U_F at $\mathbf{0}$. The sum of the solid angles of the topological components of $Y(V, E)$ is 4π .

Definition 5 (local fan). A triple (V, E, F) is a local fan if the following conditions hold.

1. (V, E) is a fan.
2. F is a face of $H = \text{hyp}(V, E)$.
3. H is isomorphic to Dih_{2k} , where $k = \text{card}(F)$.

Definition 6 (localization). Let (V, E) be a fan and let F be a face of $\text{hyp}(V, E)$. Let

$$\begin{aligned} V' &= \{\mathbf{v} \in V : \exists \mathbf{w} \in V. (\mathbf{v}, \mathbf{w}) \in F\}, \\ E' &= \{\{\mathbf{v}, \mathbf{w}\} \in E : (\mathbf{v}, \mathbf{w}) \in F\}. \end{aligned}$$

The triple (V', E', F) is called the localization of (V, E) along F .

The localization is a local fan.

3 Strategies

The strategy of the proof is to classify the hypermaps of fans $(V, E_2(V))$ for $V \in \mathcal{V}$ and to show that there are only two possibilities: the contact hypermaps of the FCC and the HCP. From this, the proof of Fejes Tóth's conjecture ensues.

The classification result is analogous to the one that we have obtained for tame hypermaps in [Hal12a]. This suggests developing a proof along exactly the same lines. We define a collection of hypermaps with properties that are analogous to those defining a tame hypermap and call them hypermaps with *tame contact*. A computer generated classification of these hypermaps gives only a few possibilities. Those other than the FCC and HCP hypermaps are eliminated by linear programming methods.

Remark 1 (Lexell's theorem). According to Lexell's theorem, for any two distinct non-antipodal points $\mathbf{u}, \mathbf{v} \in S^2(2)$, the locus of points $\mathbf{w} \in S^2(2)$, along which the spherical triangle with vertices $\mathbf{u}, \mathbf{v}, \mathbf{w}$ has given fixed area, is a circular arc with endpoints at the antipodes of \mathbf{u} and \mathbf{v} .

Lexell's theorem is an aid in finding the minimum of $\text{sol}(U_F)$. It is a consequence of Lexell's theorem that the area of a spherical triangle (viewed as a function of its edge lengths) does not have a interior point local minimum, when the edge lengths are constrained to lie in given intervals. For the spherical triangle of minimal area, each edge length is extremal.

Remark 2 (Leech's solution of the Newton–Gregory problem). During a famous discussion with Gregory, Newton asserted that if $V \subset S^2(2)$ is any packing, then $\text{card}(V) \leq 12$. That is, at most twelve nonoverlapping balls can touch a fixed central ball.

Leech's proof of Newton's assertion is noteworthy [Lee56]. Assuming the existence of a packing $V \subset S^2(2)$ of cardinality thirteen, Leech associates a planar graph (V, E) with V , which is similar to our standard fan. In our notation, he estimates the solid angle of each topological component U_F . These solid angle estimates can be verified with Lexell's theorem. Next, he classifies the planar graphs (V, E) that satisfy various combinatorial constraints obtained from the solid angle estimates. He finds that no such planar graph exists, in confirmation of Newton's assertion.

4 Main Estimate

Let (V, E) be a fan and let F be a face of $\text{hyp}(V, E)$. When $x = (\mathbf{v}, \mathbf{w})$ is a dart in $\text{hyp}(V, E)$, define $h(x) = \|\mathbf{v}\|/2$. Define the weight function

$$\tau(V, E, F) = \text{sol}(U_F) + (2 - k(F)) \text{sol}_0 \quad (2)$$

where $\text{sol}_0 = 3 \arccos(1/3) - \pi \approx 0.55$ is the solid angle of a spherical equilateral triangle with a side of arclength $\pi/3$, and $k(F)$ is the cardinality of F .

Definition 7 $(\mathcal{V}', E^+(V))$. Let V be a packing in \mathbb{R}^3 . Write $E^+(V)$ for the set of pairs $\{\mathbf{u}, \mathbf{v}\} \subset V$ such that $2 \leq \|\mathbf{u} - \mathbf{v}\| < \sqrt{8}$. Let $\mathcal{V}' \subset \mathcal{V}$ be the subset of packings $V \in \mathcal{V}$ that satisfy the following two properties.

1. $(V, E^+(V))$ is a fan.
2. The graph $(V, E^+(V))$ is biconnected.

The next theorem is the main estimate for packings with kissing number twelve.

Theorem 2 (main estimate). *Let $V' \in \mathcal{V}'$. Let F be a face of $(V', E^+(V'))$ with at least three darts, and let (V, E, F) be the localization of $(V', E^+(V'))$ along F . Let $U = U_F$ be the topological component of $Y(V, E)$ corresponding to F . Let S be the subset of E consisting of edges $\{\mathbf{v}, \mathbf{w}\}$ such that $2h_0 \leq \|\mathbf{v} - \mathbf{w}\| < \sqrt{8}$. Set $r = \text{card}(E \setminus S)$ and $s = \text{card}(S)$. Then*

$$\tau(V, E, F) \geq \min(d_2(r, s), \text{tgt}), \quad (3)$$

where

$$d_2(r, s) = \begin{cases} 0, & \text{if } (r, s) = (3, 0), \\ 0.103(2 - s) + 0.27(r + 2s - 4), & \text{otherwise.} \end{cases}$$

and $\text{tgt} = 1.541$.

Proof. We take the spherical Delaunay triangulation of the sphere $S^2(2)$ induced by V' . Triangles correspond to triangulated faces of the polyhedron obtained as the convex hull of $V' \subset \mathbb{R}^3$. By standard estimates [Hal12a], by the length constraint $\|\mathbf{v} - \mathbf{w}\| < \sqrt{8}$, each edge of $E^+(V')$ is an edge of this polyhedron and gives an edge of a Delaunay triangle. Thus U_F , up to a set of measure zero, is a disjoint union of cones over Delaunay triangles:

$$U_F \approx \bigcup \text{aff}_+^0(\mathbf{0}, \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}).$$

We may calculate the solid angle of U_F and also $\tau(U_F)$ by this triangulation.

By the kissing number problem, V' , which has cardinality 12, is a saturated spherical network on $S^2(2)$; that is, there is no room to add a further point on $S^2(2)$ that has distance at least 2 from all points of V' . By this saturation property, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subset V'$ is a Delaunay triangle, then the circumradius of the simplex $\{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is less than 2. Since $\mathbf{u}_i \in S^2(2)$, this corresponds to a Euclidean triangular circumradius less than $\sqrt{3}$ for $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

By construction, every edge in $E \setminus S$ has length 2. Edges of Delaunay triangles that are not in E have length at least $\sqrt{8}$. The upper bound on these edges will be determined by the circumradius constraint.

Fix attention on a single Delaunay triangle $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Shifting notation, let r be the number of edges of the triangle of length 2. Let s be the number of edges of length in the range $[2h_0, 3.0)$. Let t be the remaining number of edges; that is, the number of those of length at least 3.0. We have $r + s + t = 3$.

Define $d_3 : \mathbb{N}^3 \rightarrow \mathbb{R}$ by

$$d_3(r, s, t) = \begin{cases} 0, & \text{if } (r, s, t) = (3, 0, 0), \\ 0, & \text{if } (r, s, t) = (2, 0, 1) \\ 0.103(2 - s) + 0.27(r + 2s + 2t - 4), & \text{otherwise.} \end{cases}$$

Note that $d_3(r, s, 0) = d_2(r, s)$.

We claim that the solid angle $A = \text{sol}(\mathbf{0}, \text{aff}_+^0(\mathbf{0}, \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}))$ satisfies

$$A \geq \text{sol}_0 + d_3(r, s, t). \quad (4)$$

To show this, we work case by case in the parameters (r, s, t) . Suppose first that $(r, s, t) \neq (1, 1, 1)$. If $t > 0$, we use the triangle circumradius bound $\sqrt{3}$ to give an upper bound $b_{r,s,t}$ on the edges of length at least 3.0. Then we use Lexell's theorem to calculate bounds on the area of the Delaunay triangle, given the ranges on edges, and in each case we find that the area estimate is satisfied.

We work a few cases explicitly. For example if $(r, s, t) = (3, 0, 0)$, then the solid angle is exactly sol_0 and the constant d_3 is 0. The estimate is sharp in this case. If $(r, s, t) = (2, 0, 1)$, then the circumradius-derived upper bound on the long edge is $\sqrt{32/3}$, the solid angle is at least sol_0 , and the constant d_3 is 0. The estimate is sharp in this case as well. If $(r, s, t) = (0, 0, 3)$, then an upper bound on the long edges is 3.27 (the circumradius of a triangle with sides 3.0, 3.0, 3.27 is greater than $\sqrt{3}$), by Lexell the area of a spherical triangle with three (Euclidean) edges in the interval $[3.0, 3.27]$ is at least

$$\pi/2 > \text{sol}_0 + d_3 = \text{sol}_0 + 3(0.27).$$

By a simple *computer calculation*² [Hal12b], the other cases have been checked in a similar way.

In the case $(r, s, t) = (1, 1, 1)$, there is an edge length in the interval $[2h_0, 3.0)$. If the upper bound on the longest edge is at most 3.45, then the lower bound is calculated by Lexell following the procedure just described. Again, if the lower bound on the midlength edge is at least 2.6, then the procedure gives the bound. However, if both of these conditions fail, we have a triangle whose three edges fall in the intervals $[2, 2]$, $[2h_0, 2.6]$, $[3.45, 2\sqrt{3}]$, respectively. Every triangle with these edge bounds is obtuse and has circumradius at least $\sqrt{3}$. Thus, the situation is vacuous, and the bound (4) holds in the case $(r, s, t) = (1, 1, 1)$.

Next, we observe that the function d_3 is superadditive. If we combine two adjacent regions with parameters (r_1, s_1, t_1) and (r_2, s_2, t_2) , the combined region has parameters (r', s', t') , where

$$(r', s', t') = \begin{cases} (r_1 + r_2, s_1 + s_2, t_1 + t_2 - 2) & \text{if the shared edge has length at least 3.0} \\ (r_1 + r_2, s_1 + s_2 - 2, t_1 + t_2) & \text{if the shared edge has length in } [2h_0, 3.0). \end{cases}$$

Then we can verify by a routine case-by-case calculation that

$$d_3(r_1, s_1, t_1) + d_3(r_2, s_2, t_2) \geq d_3(r', s', t').$$

(Note that $(r_i, s_i, t_i) \neq (3, 0, 0)$ and that we cannot have $(r_1, s_1, t_1) = (r_2, s_2, t_2) = (2, 0, 1)$ because a quadrilateral of side length 2 always has a diagonal of length at most $\sqrt{8}$.)

To complete the proof, we show that the bound (4) gives the bound of the lemma. Let A_i , for $i = 1, \dots, (k-2)$, be the areas of the Delaunay triangles in the partition of U_F .

² [HFBBNUL] This was done in Mathematica. Footnotes give randomly assigned tracking numbers that can be used to locate the details of the calculation at the website [Hal12b].

We write (r_i, s_i, t_i) for the parameters of A_i and return to the earlier notation of (r, s) as the parameters defined in the statement of the lemma. By superadditivity, we have

$$\begin{aligned}
\tau(U_F) &= (2 - k(F)) \text{sol}_0 + \text{sol}(U_F) \\
&= (2 - k(F)) \text{sol}_0 + \sum_i A_i \\
&= \sum_i (-\text{sol}_0 + A_i) \\
&\geq \sum_i d_3(r_i, s_i, t_i) \\
&\geq d_3(r, s, 0) \\
&= d_2(r, s).
\end{aligned}$$

The lemma ensues. □

5 Biconnected Fans

We may create fans that are biconnected graphs in the same way as in [HF06]. Here is a review of the construction.

Lemma 3. *Let $V \in \mathcal{V}$. Then there exists $V' \in \mathcal{V}'$ and a bijection $\phi : V' \mapsto V$ that induces a bijection of contact graphs:*

$$\phi_* : (V, E_2(V)) \cong (V', E_2(V')).$$

Proof. Begin with the fan $(V, E_2(V))$.

We claim that $(V, E^+(V))$ is a fan. Indeed, it is checked by [HF06, Lemma 4.30] that the blades satisfy the intersection property of fans, except possibly when two new blades are the diagonals of a quadrilateral face in $(V, E_2(V))$. (The cited lemma uses the constant 2.51 instead of $2h_0$, but this does not affect the reasoning of the lemma.) We may directly rule out the possibility of a quadrilateral face as follows. The diagonals of a quadrilateral face in $(V, E_2(V))$ is a spherical rhombus and one of its diagonals is necessarily at least $\sqrt{8}$ (with extreme case a square of side 2). The other fan properties are easily checked.

If the hypermap $\text{hyp}(V, E^+(V))$ is not connected, the set of nodes $V_1 \subset V$ in one combinatorial component can be moved closer to another combinatorial component until a new edge is formed. This can be done in a way that the deformation of V remains in \mathcal{V} and no new edges of length at most $2h_0$ are formed. Continuing in this fashion, a connected hypermap is obtained.

A biconnected hypermap is produced by further deformations of the fan around each *articulation node* (that is, a node whose deletion increases the number of combinatorial components). □

Definition 8 ($D_U, m_U, r_U, s_U, k_U, \tau(U)$). *Let $V \in \mathcal{V}$. Let U be a topological component of $Y(V, E_2(V))$ and let D_U be the set of all darts of $\text{hyp}(V, E_2(V))$ that lead into U . For*

each $x \in D_U$, let $m(x) > 0$ be the smallest positive integer such that $f^m x$ and x lie at the same node. Let m_U be the maximum of $m(x)$ as x runs over D_U . The constant m_U can be viewed as a simplified face size. Let r_U be the number of nonisolated darts in D_U , and let $2 + s_U$ be twice the number of combinatorial components of $\text{hyp}(V, E_2(V))$ that meet D_U . (In particular, s_U is even and $s_U = 0$ exactly when D_U lies in a single combinatorial component of the hypermap.) Let $k_U = r_U + s_U$. Overloading the symbol τ , we set $\tau(U) = \text{sol}(U) + (2 - k_U) \text{sol}_0$. (If a single face F leads into U and if the face is simple, then the overloaded notation is consistent with the earlier notation: $\tau(U) = \tau(V, E_2(V), F)$.)

Lemma 4. *Let $V \in \mathcal{V}'$. Let U be a topological component of $Y(V, E_2(V))$. Then $\tau(U) \geq \min(d_2(r_U, s_U), \text{tgt})$.*

Proof. Up to a null set (given by the finite union of blades $C^0(\varepsilon)$ for $\varepsilon \in E^+(V) \setminus E_2(V)$), the region U is the union of topological components U_F of $Y(V, E^+(V))$, which are in bijection with the faces F of $\text{hyp}(V, E^+(V))$. The function $\tau(U)$ is additive:

$$\tau(U) = \sum_{U_F \subset U} \tau(V, E^+(V), F). \quad (5)$$

By the biconnectedness of $(V, E^+(V))$, each value $\tau(V, E^+(V), F)$ is the same before and after localization. (Localization replaces $(V, E^+(V), F)$ with (V', E', F) where $V' \subset V$, $E' \subset E^+(V)$, and (V', E', F) is a local fan.) Lemma 2 gives a lower bound on the constants $\tau(V, E^+(V), F)$. The constants $d_2(r_U, s_U)$ are superadditive:

$$d_2(r_U, s_U) \leq \sum_{U_F \subset U} d_2(r(V, F), s(V, F)),$$

where $s(V, F)$ is the cardinality of the set of edges of $E^+(V) \setminus E_2(V)$ that meet F , and $r(V, F) = \text{card}(F) - s(V, F)$. Thus, the lower bound on $\tau(U)$ follows from the main estimate (Theorem 2). \square

Lemma 5. *Let $V \in \mathcal{V}'$. Then*

$$\sum_{U \in [Y(V, E_2(V))]} \tau(U) = (4\pi - 20 \text{sol}_0),$$

where $[Y]$ is the set of topological components of $Y = Y(V, E_2(V))$.

Proof. For a packing of twelve points $V \subset S^2(2)$, we have $12 = \mathcal{L}(V)$, where $\mathcal{L}(V)$ is the left-hand side of equation (1). From this equality, following [Hal12a, 8.2.3], we have

$$\sum_F \tau(V, E^+(V), F) = (4\pi - 20 \text{sol}_0),$$

the sum running over faces of $\text{hyp}(V, E^+(V))$. The result follows by additivity (5). \square

The constant $\text{tgt} = 1.541$ is slightly larger than $(4\pi - 20 \text{sol}_0) \approx 1.54065$. The constants $d_2(r_U, s_U)$ are nonnegative, so that $\tau(U)$ is as well. This means that for every subset A of $[Y(V, E_2(V))]$, we have

$$\sum_{U \in A} \tau(U) < \text{tgt}. \quad (6)$$

Lemma 6 (biconnected). *Let $V \in \mathcal{V}$. Then $\text{hyp}(V, E_2(V))$ is biconnected.*

Proof. By Lemma 3, we may replace V with a new set in \mathcal{V} if necessary so that $(V, E^+(V))$ is a biconnected fan. We show that the smaller fan $(V, E_2(V))$ is also biconnected.

Let U be a topological component of $Y(V, E_2(V))$. Lemma 4 implies that $\tau(U) \geq \min(d_2(r_U, s_U), \text{tgt})$.

We claim that if $m_U \leq 5$, then D_U is a simple face. Otherwise, either D_U is a face that is not simple, or it consists of more than one face. Either way, some node \mathbf{v} lies in the interior to the m_U -gon. Let \mathbf{u}, \mathbf{w} be consecutive nodes around the m_U -gon. By a *computer calculation*³ [Hal12b] the angles $\text{azim}(\mathbf{0}, \mathbf{v}, \mathbf{u}, \mathbf{w})$ are each less than $2\pi/5$. The angles around \mathbf{v} cannot sum to 2π as required.

We claim that D_U is a simple face. Otherwise, assume for a contradiction that D_U is not simple, $m_U \geq 6$, and $d_2(r_U, s_U) < \text{tgt}$. From the classification of [HF06, p. 126, Fig. 12.1], and the inequalities $d_2(9, 0) > \text{tgt}$, $d_2(6, 2) > \text{tgt}$, it follows that $m_U = 6$ and $\tau(U) \geq d_2(8, 0)$. The set D_U meets seven nodes: the six nodes counted by m_U and a node in the interior of the hexagonal arrangement. At each node there is a face of the hypermap $\text{hyp}(V, E_2(V))$ that is not a triangle, because 2π is not an integer multiple of the dihedral angle of a regular tetrahedron. As the packing has twelve nodes in all, five nodes remain, each meeting a nontriangular topological component of $Y(V, E_2(V))$. Thus, by counting nodes, the hypermap has at least one pentagon or two quadrilaterals. We find that $\sum_U \tau(U)$ is at least

$$d_2(8, 0) + d_2(5, 0) > \text{tgt}, \quad \text{or} \quad d_2(8, 0) + 2d_2(4, 0) > \text{tgt},$$

which is contrary to (6). Thus, D_U is a simple face.

We claim that the hypermap is biconnected. Otherwise, if the hypermap is not connected, then we can find two faces of the hypermap that lead into the same topological component of $Y(V, E_2(V))$. If the hypermap is connected but not biconnected, then some face of the hypermap is not simple. Both possibilities contradict the fact that D_U is a simple face. \square

6 Tame Contact

This subsection defines a notion of tameness that includes hypermaps that arise as the fan of a packing with kissing number twelve. In the definition of tame hypermap in [Hal12a], a function b is used. In this section we use a similar function, which is again called b . Recall that $\text{tgt} = 1.541$.

Definition 9 (b). *Define $b : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ by $b(p, q) = \text{tgt}$, except for the following values:*

$$b(0, 3) = b(1, 3) = 0.618, \quad b(2, 2) = 0.412.$$

³ [6621965370] (Mathematica)

Definition 10 (d). Define $d_1 : \mathbb{N} \rightarrow \mathbb{R}$ by

$$d_1(k) = \begin{cases} 0, & k \leq 3, \\ 0.206 + 0.27(k - 4), & k = 4, \dots, 8, \\ \text{tgt}, & k > 8. \end{cases}$$

The function d_1 is related to the two-variable function in Lemma 2: $d_1(k) = d_2(k, 0)$, when $4 \leq k \leq 8$.

We say that a node of a fan (V, E) has type $(p, q, r) \in \mathbb{N}^3$ if at the node there are $p + q + r$ faces, of which p are triangles and q are quadrilaterals.

Definition 11 (weight assignment). A weight assignment on a hypermap H is a function τ on the set of faces of H taking values in the set of nonnegative real numbers. A weight assignment τ is a contact weight assignment if the following two properties hold.

1. If the face F has cardinality k , then $\tau(F) \geq d_1(k)$.
2. If a node v has type $(p, q, 0)$, then

$$\sum_{F: v \cap F \neq \emptyset} \tau(F) \geq b(p, q).$$

The sum $\sum_F \tau(F)$ is called the total weight of τ .

Definition 12 (tame contact). A hypermap has tame contact if it satisfies the following conditions.

1. (BICONNECTED) The hypermap is biconnected. In particular, each face meets each node in at most one dart.
2. (PLANAR) The hypermap is plain and planar. (A hypermap is defined to be plain when e is an involution on D ; that is, $e^2 = I_D$. A connected hypermap is planar when Euler's relation holds: $\#n + \#e + \#f = \text{card}(D) + 2$, where for any permutation h , we write $\#h$ for the number of orbits of h on D .)
3. (NONDEGENERATE) The edge map e has no fixed points.
4. (NO LOOPS) The two darts of each edge lie in different nodes.
5. (NO DOUBLE JOIN) At most one edge meets any two given nodes.
6. (FACE COUNT) The hypermap has at least two faces.
7. (FACE SIZE) The cardinality of each face is at least three and at most eight.
8. (NODE COUNT) The hypermap has twelve nodes.
9. (NODE SIZE) The cardinality of every node is at least two and at most four.
10. (WEIGHTS) There exists a contact weight assignment of total weight less than tgt .

Theorem 3. The contact hypermap $\text{hyp}(V, E_2(V))$ of a packing $V \in \mathcal{V}$ is a hypermap with tame contact.

Proof. It is enough to go through the list of properties that define a tame contact hypermap and to verify that the contact hypermap satisfies each one. We use the weight assignment $F \mapsto \tau(V, E_2(V), F)$.

1. (BICONNECTED) The hypermap is biconnected by Lemma 6.
2. (PLANAR) The contact hypermap is plain and planar by the general properties of fans. (See [Hal12a, Lemma 5.8] and [Hal12a, 5.3].)
3. (NONDEGENERATE) The edge map has no fixed points by the general properties of fans. See [Hal12a, Lemma 5.8].
4. (NO LOOPS) There are no loops by the general properties of fans. See [Hal12a, Lemma 5.8].
5. (NO DOUBLE JOIN) This is also a general property of fans. See [Hal12a, Lemma 5.8].
6. (FACE COUNT) Each node has at least two darts by biconnectedness. Each face is simple; so the two darts at a node lie in different faces. Thus, the hypermap has at least two faces.
7. (FACE SIZE) The cardinality of each face is at least three because, as we have just observed, the hypermap has no loops or double joins. The cardinality of a face is at most eight because of the estimate $d_2(9, 0) > \text{tgt}$.
8. (NODE COUNT) There are twelve nodes by the definition of a packing with kissing number twelve.
9. (NODE SIZE) We have already established that the cardinality of each node is at least two. The proof that the cardinality is never greater than four appears in Lemma 7.
10. (WEIGHTS) Theorem 2 establishes the inequality $\tau(V, E_2(V), F) \geq d_1(k)$. The total weight of the weight assignment is given by Lemma 5:

$$\sum_F \tau(V, E_2(V), F) = (4\pi - 20 \text{ sol}_0) < \text{tgt}.$$

Let \mathbf{v} be a node of type $(p, q, 0)$. Then by the main estimate,

$$\sum_{F : F \cap \mathbf{v} \neq \emptyset} \tau(V, E_2(V), F) > d_1(4) q.$$

This gives the nonzero entries in the table of bounds $b(p, q)$. The remaining entries follow from Lemma 7.

□

Lemma 7. *Let $V \in \mathcal{V}$. Every node of $(V, E_2(V))$ has degree at most four. Furthermore, suppose the type of a node is $(p, q, 0)$. Then (p, q) must be*

$$(0, 3), (1, 3), \text{ or } (2, 2).$$

Proof. The interior angles of a spherical polygon in the contact graph have the following lower α_k and upper bounds β_k , as a function of the number of sides k .

k	α_k	β_k	
3	$\text{dih}(2, 2, 2, 2, 2, 2)$	$\text{dih}(2, 2, 2, 2, 2, 2)$	(7)
4	$\text{dih}(2, 2, 2, 2h_0, 2, 2)$	$2 \text{ dih}(2, 2, 2, 2, 2h_0, 2)$	
≥ 5	$\text{dih}(2, 2, 2, 2h_0, 2, 2)$	2π .	

Thus,

$$p\alpha_3 + q\alpha_4 + r\alpha_5 \leq 2\pi \leq p\beta_3 + q\beta_4 + r\beta_5.$$

There are no solutions for (p, q, r) in natural numbers when $p + q + r \geq 5$ and only the three given solutions in (p, q, r) with $r = 0$. □

7 Classification

The website for the computer code contains a list of eight hypermaps that have been obtained by running the classification algorithm with the tame contact parameters [Hal12b].

Lemma 8 (tame hypermap classification). *Every hypermap with tame contact is isomorphic to a hypermap in the given list of eight hypermaps, or is isomorphic to the opposite of a hypermap in the list.*

Proof. By a computer calculation⁴ [Hal12b], the set of all hypermaps has been classified by the same algorithm described in [Hal12a, Chapter 4]. \square

Lemma 9. *Let $V \in \mathcal{V}$. Suppose that $H = \text{hyp}(V, E_2(V))$ is a hypermap with tame contact. Then H is the FCC or HCP contact hypermap.*

Proof. The explicit enumeration of hypermaps with tame contact has eight cases. Two are the hypermaps of the FCC and HCP. The remaining six must be eliminated. A geometrical argument eliminates one of these cases and linear programming eliminates the other five.

We claim that one case with a hexagon cannot be realized geometrically as a contact fan (Figure 1). Indeed, the perimeter of a hexagon with sides $\pi/3$ is 2π . However, the hexagons are geodesically convex, and 2π is a strict upper bound on the perimeter of the hexagon. Thus, this case does not exist.

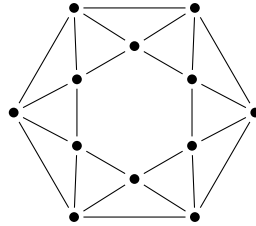


Fig. 1. This hypermap has tame contact but cannot be realized as a packing in \mathcal{V} .

There are some linear programming constraints that are immediately available to us.

1. The angles around each node sum to 2π .
2. Each angle of a triangle is α_3 .
3. Each angle of each rhombus lies between α_4 and β_4 .
4. The opposite angles of each rhombus are equal.

By a linear programming computer calculation⁵ [Hal12b], these systems of constraints are infeasible in the remaining five cases. \square

⁴ [PYWHMHQ]

⁵ [JKJNYAA]

Lemma 10. *Let $V \in \mathcal{V}$ be a packing such that $\text{hyp}(V, E_2(V))$ is isomorphic to the FCC or HCP contact hypermap. Then V is congruent to the FCC or HCP configuration in $S^2(2)$.*

Proof. Every face of the hypermap of $(V, E_2(V))$ is a triangle or quadrilateral. The eight triangles in the FCC or HCP contact hypermap determine eight equilateral triangles in V of edge length 2. The eight triangles rigidly determine V up to congruence. \square

Proof (Proof of Theorem 1). The contact hypermap of a packing with kissing number twelve has tame contact. By Theorem 9, this hypermap is that of the FCC or HCP. By Lemma 10, the kissing configuration of the packing is congruent to the FCC or HCP. As the center of the packing may be chosen at an arbitrary point in the packing, every point in the packing is congruent to one of these two arrangements. The result ensues. \square

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